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Question 1

a) $\vec{v} = (\cos\theta / r^2)\,\hat{r} + (r/\sin\theta)\,\hat{\theta} + (r\cos\theta\,\sin\phi)\,\hat{\phi}$ – in spherical coordinates

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\cdot\frac{\cos\theta}{\cancel{r^2}}\right) + \frac{1}{r\cdot\sin\theta}\frac{\partial}{\partial\theta}\left(\cancel{\sin\theta}\cdot\frac{r}{\cancel{\sin\theta}}\right) + \frac{1}{\cancel{r}\cdot\sin\theta}\frac{\partial}{\partial\phi}\left(\cancel{r}\cdot\cos\theta\cdot\sin\phi\right)$$

$$= \frac{\cos\theta}{r^2} + \frac{1}{\sin\theta} + \frac{\cos\theta\cdot\cos\phi}{\sin\theta} = \frac{\sin\theta\cdot\cos\theta + r^2 + r^2\cdot\cos\theta\cdot\cos\phi}{r^2\sin\theta}$$

b) $\vec{v} = s(2+\sin^2\phi)\,\hat{s} + s\cdot\sin\phi\cos\phi\,\hat{\phi} + 3sz\phi\,\hat{z}$ – in cylindrical coordinates

$$\vec{\nabla}\times\vec{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s\,v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{z}$$

$$= \left[\frac{1}{s}\cdot 3sz - 0\right]\hat{s} + \left[0 - 3z\phi\right]\hat{\phi} + \frac{1}{s}\left[2\cdot s\cdot\sin\phi\cos\phi - s\cdot 2\cdot\sin\phi\cos\phi\right]\hat{z}$$

$$= 3z\,\hat{s} - 3z\phi\,\hat{\phi} + 0\,\hat{z} = 3z\,\hat{s} - 3z\phi\,\hat{\phi}$$

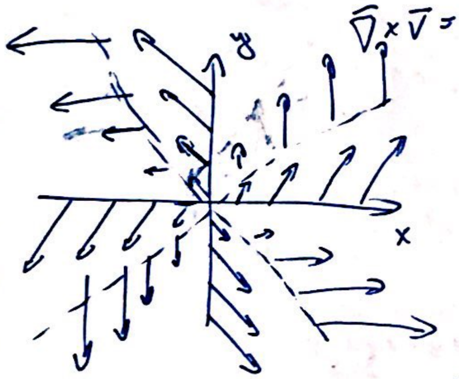
c) $\vec{\nabla}\cdot\vec{v} > 0$

$\vec{\nabla}\times\vec{v} \neq 0$

[$\vec{v} = (x-y)\,\hat{x} + (x+y)\,\hat{y}$

$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(x+y) = +1+1 = 2 > 0$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & x+y & 0 \end{vmatrix} = \hat{x}(0) + \hat{y}(0) + \hat{z}(1-(-1)) = 2\hat{z} \neq 0$$



d) 

$$V = \int_V d\tau = \int_0^{2\pi}\int_0^{\pi}\int_a^{2a} r^2\cdot\sin\theta\,dr\,d\theta\,d\phi = \int_a^{2a} r^2\,dr\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}\phi =$$

$$= \frac{r^3}{3}\Big|_{r=a}^{2a}(-\cos\theta)\Big|_{\theta=0}^{\pi}\,\phi\Big|_{\phi=0}^{2\pi} = \left(\frac{8a^3}{3} - \frac{a^3}{3}\right)(1+1)\,2\pi = 4\pi\,\frac{7a^3}{3}$$

in spherical: $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$r: a \to 2a$

$\theta: 0 \to \pi$

$\phi: 0 \to 2\pi$

$d\vec{a} = r^2\sin\theta\,d\theta\,d\phi$

$$S = \underset{r=a}{S_{inner}} + \underset{r=2a}{S_{outer}} = \iint_{r=a} r^2\cdot\sin\theta\,d\theta\,d\phi + \iint_{r=2a} r^2\cdot\sin\theta\,d\theta\,d\phi$$

$$= a^2\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi + 4a^2\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi = 5a^2\left[(-\cos\theta)\Big|_0^{\pi}\phi\Big|_0^{2\pi}\right]$$

$$= 5a^2\cdot 2\cdot 2\pi = 20\pi a^2$$

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Question 2

charge enclosed by surface

a) $\Phi_E = \oint_{S_1} \vec{E}\cdot d\vec{a} = \frac{Q_{enc1}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$ – From Gauss's law : Flux is directly proportional to charge enclosed by a surface.

$\Phi_2 = ? = \oint_{S_2} \vec{E}\cdot d\vec{a} = \frac{Q_{enc2}}{\varepsilon_0} = \frac{q+2q}{\varepsilon_0} = \frac{3q}{\varepsilon_0} = \underline{\underline{3\Phi_E}}$

S_1 encloses $+1q$ charge, S_2 encloses $+3q$ charge

$\Rightarrow \Phi_2 = 3\Phi_1 = 3\Phi_E$

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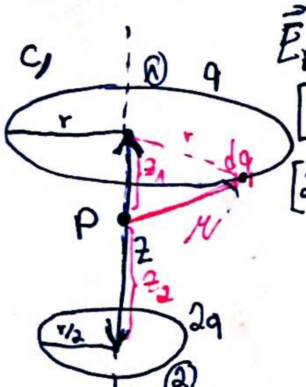
b) $\vec{V} = \frac{1}{2r^2}\hat{r}$

$\vec{\nabla}\cdot\vec{V} = \vec{\nabla}\cdot\left(\frac{\hat{r}}{2r^2}\right) = \frac{1}{2}\left[\underbrace{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)}_{=4\pi\delta^3(\vec{r})}\right] = \frac{1}{2}\cdot 4\pi\delta^3(\vec{r}) = \underline{\underline{2\pi\delta^3(\vec{r})}}$

c)



$\vec{E}_p = ?$

$[q] = C\cdot m^{-1}$ $\left(\begin{array}{l} q_1 = q\cdot 2\pi r \text{ ---total charge on larger loop} \\ q_2 = 2q \text{ ---- total charge on smaller loop}\end{array}\right)$

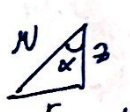
$[2q] = C$

The ~~two~~ electric fields of larger and smaller loop ~~~~ (~~vector~~ like vectors), therefore we add up ~~therefor~~ can calculate for ~~ea~~ each loop separately and then add it.

Each point on a circle has a point on the opposite side of it and since the loop carries uniform charge, the horizontal components will cancel out, leaving only the vertical (z) component.

$\Rightarrow$ For point charge $\underline{dq}$ ~~on~~ ~~the~~ a loop: $E_z = |E|\cos\alpha = \frac{1}{4\pi\varepsilon_0}\frac{dq}{N^2}\cos\alpha$

$= \frac{1}{4\pi\varepsilon_0}\frac{dq}{z^2+r^2}\cdot\frac{z}{\sqrt{z^2+r^2}}$



$\cos\alpha = \frac{z}{N}$

$N = \sqrt{z^2+r^2}$

Loop is a sum of infinitely many dq charges from above, therefore:

$\vec{E} = \int \frac{1}{4\pi\varepsilon_0}\frac{z}{(z^2+r^2)^{3/2}}\,dq\,\hat{z} =$

For upper loop:

$dq = \lambda_1 dl_\phi = \lambda_1 (r\cdot\sin\theta)\,d\phi$

$r = r,\ \theta = \frac{\pi}{2},\ \phi: 0 \to 2\pi$

For lower loop:

$dq = \lambda_2 (r\cdot\sin\theta)\,d\phi$

$r = \frac{r}{2},\ \theta = \frac{\pi}{2},\ \phi: 0 \to 2\pi$

$dq = \lambda_2\left(\frac{r}{2}\cdot\sin\theta\right)d\phi$

$= \frac{\lambda_1}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{z\cdot r\cdot\sin\theta}{(z^2+r^2)^{3/2}}\,d\phi\,\hat{z}$

$\vec{E}_1 = \frac{\lambda_1}{4\pi\varepsilon_0}\frac{2\pi z_1 r}{(z_1^2+r^2)^{3/2}}\hat{z}$

$\vec{E}_2 = \frac{\lambda_2}{4\pi\varepsilon_0}\frac{\pi\cdot z_2\cdot r}{\left(z_2^2+\frac{r^2}{4}\right)^{3/2}}\hat{z}$

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$$\vec{E}_1 = -\frac{q}{4\pi\varepsilon_0} \frac{2\pi z_1 r}{(z_1^2 + r^2)^{3/2}} \hat{z} \qquad \lambda_2 = \frac{\text{total charge}}{\text{length}} = \frac{\cancel{2}q}{\cancel{2}\pi \frac{r}{2}} = \frac{2q}{\pi r}$$

$$\vec{E}_2 = \frac{2q}{4\pi\varepsilon_0} \frac{\cancel{2\pi} z_2 \cancel{r}}{\cancel{\pi}\cancel{r}(z_2^2 + \frac{r^2}{4})^{3/2}} \hat{z} \qquad z_2 = z - z_1$$

$$\boxed{\text{Between loops points } \vec{E}_1 \text{ in } -\hat{z} \text{ and } \vec{E}_2 \text{ in } +\hat{z}}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{q}{4\pi\varepsilon_0}\left(-\frac{2\pi z_1 r}{(z_1^2 + r^2)^{3/2}} + \frac{2(z - z_1)}{((z - z_1)^2 + \frac{r^2}{4})^{3/2}}\right)\hat{z}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left(-\frac{2\pi z_1 r}{(z_1^2 + r^2)^{3/2}} + \frac{2z - 2z_1}{(z^2 - 2zz_1 + z_1^2 + \frac{r^2}{4})^{3/2}}\right)\hat{z}$$

where r is radius of larger loop
z is separation of the loops
z_1 is distance of P from the larger loop (to ~~set~~ define position)

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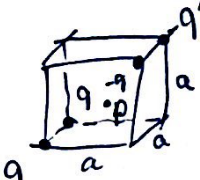
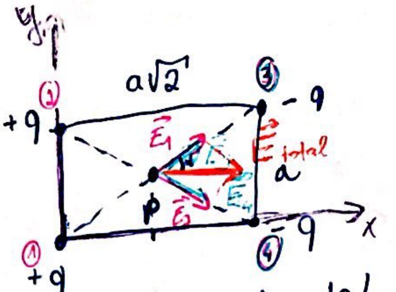
Question 3

a) $d = a$

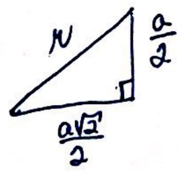


all charges and center of cube lie in a plane



$\cancel{\frac{a}{a\sqrt{3}} = \frac{\sqrt{3}}{2}}$

$\vec{E}_{tot} = ?$



$\Rightarrow |r| = \sqrt{\frac{a^2}{4} + \frac{2a^2}{4}} = a\sqrt{\frac{3}{4}} = \frac{a}{2}\sqrt{3}$

$|\vec{E}_1| = |\vec{E}_2| = |\vec{E}_3| = |\vec{E}_4| = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{\frac{a^2}{4}\cdot 3} = \frac{1}{\pi\varepsilon_0}\frac{|q|}{3a^2}$

$|r| = |\vec{r}_1| = |\vec{r}_2| = |\vec{r}_3| = |\vec{r}_4|$

$\vec{E}_p = \vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 \quad = \vec{E}$

$= |\vec{E}_1|\hat{r}_1 + |\vec{E}_2|\hat{r}_2 - |\vec{E}_3|\hat{r}_3 - |\vec{E}_4|\hat{r}_4 = |\vec{E}|(2\hat{r}_1 + 2\hat{r}_2) = 2|\vec{E}|(\hat{r}_1 + \hat{r}_2)$

negative charges

$\boxed{\begin{matrix}\hat{r}_1 = -\hat{r}_3 \\ \hat{r}_2 = -\hat{r}_4\end{matrix}}$

$\vec{r}_1 = \frac{a\sqrt{2}}{2}\hat{x} + \frac{a}{2}\hat{y}$

$\vec{r}_2 = \frac{a\sqrt{2}}{2}\hat{x} - \frac{a}{2}\hat{y}$

$|r| = \frac{a}{2}\sqrt{3}$

$= 2|\vec{E}|\left(\frac{\hat{x}\cdot\frac{a\sqrt{2}}{2}2}{|r|}\right)$

$= 2|\vec{E}|\left(\frac{a\sqrt{2}}{|r|}\hat{x}\right)$

$= 2|\vec{E}|\left(\frac{a\sqrt{2}}{\frac{a}{2}\sqrt{3}}\right)\hat{x}$

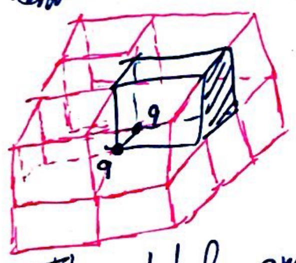
$= 2|\vec{E}|\frac{2\sqrt{2}}{\sqrt{3}}\hat{x}$

$\vec{E} = 2\cdot\frac{1}{\varepsilon_0\pi}\frac{|q|}{3a^2}\frac{2\sqrt{2}}{\sqrt{3}}\hat{x} = \frac{1}{\pi\varepsilon_0}\frac{|q|}{a^2}\frac{4\sqrt{2}}{3\sqrt{3}}\hat{x}$

↓ different approach on p.3

b) $Q_{enc} = 2q$

To use Gauss's law $\Phi_E = \oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$ we enclose the charges in a surface created by 12 cubes as shown on the left.



Let the area of the shaded side be A_0.

The total area of the surface enclosing the charges (12 cubes connected) is the area is that of $6+4+6+4+6+6 = 32$ sides of area A.

Total flux through the surface of these cubes $\Phi_{tot} = \oint 32\vec{E}\cdot d\vec{a} = \frac{2q}{\varepsilon_0}$

Therefore the flux through the shaded side follows as:

$\Phi_{tot} = 32\oint \vec{E}\cdot d\vec{a} = 32\,\Phi_{shaded} = \frac{Q_{enc}}{\varepsilon_0} = \frac{2q}{\varepsilon_0}$

$\Phi_{shaded} = \frac{2q}{32\varepsilon_0} = \frac{q}{16\varepsilon_0}$

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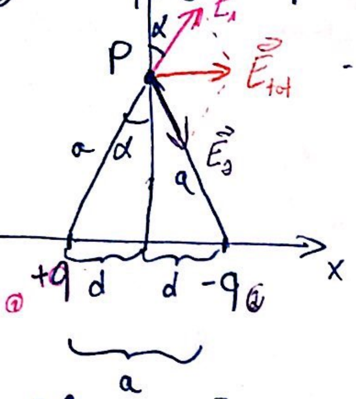
c) $\vec{E}_p = ?$



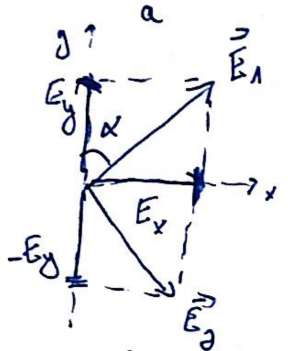
$$|\vec{E}_1| = |\vec{E}_2| = |\vec{E}| = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{a^2}$$

- in x-direction

$$\vec{E}_p = \vec{E}_1 + \vec{E}_2$$

$$\sin\alpha = \frac{d}{a} = \frac{\frac{a}{2}}{a} = \frac{1}{2} \Rightarrow \alpha = 30^\circ$$

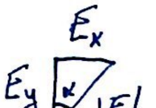
From picture on left we see that the y-components of $\vec{E}_1$ and $\vec{E}_2$ cancel out and the x-components are added together.



Hence: $|\vec{E}_{tot}| = 2E_x$

$$E_x = |\vec{E}|\cdot\sin\alpha = \frac{1}{2}|\vec{E}|$$

$$\Rightarrow |\vec{E}_{tot}| = 2\cdot\frac{1}{2}|\vec{E}| = \underline{\underline{|\vec{E}|}}$$



$$\sin\alpha = \frac{E_x}{|E|}$$

The electric field at point P will have magnitude equal to magnitude of E_1 (or E_2) and ~~be in~~ direction of +x-axis

$$\Rightarrow \underline{\underline{\vec{E}_p = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{a^2}\hat{x}}}$$

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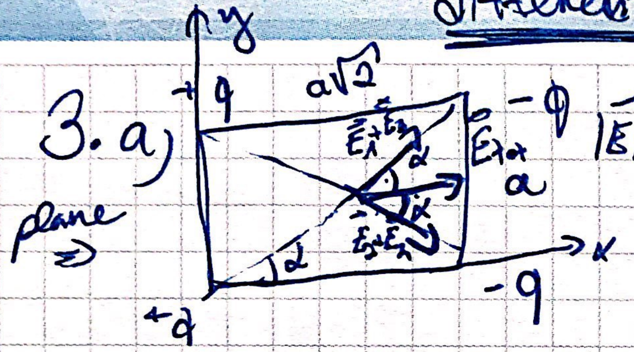
slightly different approach:

3. a)



plane ⇒

$$|\vec{E}| = |\vec{E}_1| = |\vec{E}_2| = |\vec{E}_3| = |\vec{E}_4| = \frac{1}{\pi\varepsilon_0}\frac{|q|}{3a^2}$$

y-components cancel out, only x-components left

$$E_x = |\vec{E}|\cos\alpha$$

$$\cos\alpha = \frac{a\sqrt{2}}{\sqrt{a^2+2a^2}} = \frac{a\sqrt{2}}{a\sqrt{3}}$$

$$\Longrightarrow \vec{E}_{total} = 4E_x\hat{x} = 4\frac{1}{\pi\varepsilon_0}\frac{|q|}{3a^2}\frac{\sqrt{2}}{\sqrt{3}}\hat{x}$$

$$= \frac{1}{\pi\varepsilon_0}\frac{|q|}{a^2}\frac{4\sqrt{2}}{3\sqrt{3}}\hat{x}$$

same result ✓

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